Extra Practice Problems 5

Here are some more review problems you can use to practice for the midterm. We'll release solutions on Monday, along with one final set of practice problems.

Problem One: Functions

Let $f : A \to B$ and $g : B \to C$ be functions. Notice that f's codomain is B, which happens to be the domain of C. That means that we can apply g to the output of f.

Let's define the composition of f and g as follows. If $f : A \to B$ and $g : B \to C$ are functions, the ***composition*** of f and g, denoted $(g \circ f)$, is a function $g \circ f : A \to C$ such that $(g \circ f)(x) = g(f(x))$. In other words, the function named $g \circ f$ is evaluated by applying f to the input, then applying g to the result of f.

i. Prove that if f *is* surjective and g is *not* a bijection, then $g \circ f$ is *not* a bijection.

ii. Prove that if f is *not* a bijection and g *is* injective, then $g \circ f$ is *not* a bijection.

iii. Find examples of functions f and g where neither f nor g is bijective, but $g \circ f$ is a bijection.

As hints for each of the above problems: we highly recommend starting off by drawing pictures to get a sense for how these properties interact, then turning some of your insights from those pictures into actual proofs.

Problem Two: Cardinality

A *Pythagorean triple* is a triple of positive natural numbers (a, b, c) where $a^2 + b^2 = c^2$. For example, the triple $(3, 4, 5)$ is a Pythagorean triple because $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

Let $PT = \{(a, b, c) \mid (a, b, c)$ is a Pythagorean triple.

i. Prove that $|PT| = |\mathbb{N}|$. *(Hint: Use the Cantor-Bernstein-Schroeder theorem and one of the results from Problem Set Four. Also, what happens if you multiply each number in a Pythagorean triple by the same amount?)*

We say that a set A is a ***strict subset*** of a set B, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.

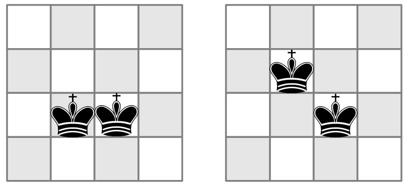
ii. Prove or disprove: If $A \subset B$, then $|A| < |B|$.

Consider the set $\mathbb{N} \cup \{\star\}$, where $\star$ is some object that isn't a natural number.

iii. Prove or disprove: $|\mathbb{N} \cup \{\star\}| = |\mathbb{N}|$.

Problem Three: A Clash of Kings

Chess is a game played on an 8×8 grid with a variety of pieces. In chess, no two king pieces can ever occupy two squares that are immediately adjacent to one another horizontally, vertically, or diagonally. For example, the following positions are illegal:



Prove that it is impossible to legally place 17 kings onto a chessboard. *(Hint: First, find a way to show that you can legally place 16 kings onto a chessboard.)*

Problem Four: Coloring a Grid, Take Two

Suppose every point in a 3×7 grid is colored either red or blue. Prove that there must be four points of the same color that form a rectangle.

Problem Five: DFAs, NFAs, and Regular Expressions

Here's a slightly tricky set of DFA, NFA, and regex design problems.

Let $\Sigma = \{ a, b \}$ and let $L = \{ w \in \Sigma^* \mid$ the length of w is a multiple of four and w contains an even number of b's $\}$.

i. Design a DFA for L.

ii. Design an NFA for the *complement* of L.

iii. Write a regular expression for the *complement* of L.

Problem Six: Nonregular Languages

A ***tautonym*** is a string formed by repeating the same string twice. For example, *dikdik* is a tautonym, as is *hotshots*. Let $\Sigma = \{ a, b \}$ and let $L = \{ ww \mid w \in \Sigma^* \}$ be the set of all tautonyms. Prove that L is not a regular language.

Problem Seven: Context-Free Grammars

Write a context-free grammar for the language $\{ a^n b^n c^m d^m \mid m, n \in \mathbb{N} \}$.